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Semiclassical universality of parametric spectral correlations

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Abstract
We consider quantum systems with a chaotic classical limit that depends on an external parameter, and study correlations between the spectra at different parameter values. In particular, we consider the parametric spectral form factor $K(\tau, x)$ which depends on a scaled parameter difference x. For parameter variations that do not change the symmetry of the system we show by using semiclassical periodic orbit expansions that the small τ expansion of the form factor agrees with random matrix theory for systems with and without time reversal symmetry.

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1. Introduction

One of the characteristic features of quantum systems with underlying chaotic dynamics lies in statistical fluctuations of their spectra. If the energy levels are scaled such that their mean separation is 1 then the statistical distribution of the levels of individual quantum chaotic systems are found to be universal in the semiclassical limit $\hbar \to 0$ and to agree with those of eigenvalues of random matrices [1]. Systems with or without time reversal symmetry (TRS) are described by the Gaussian orthogonal ensemble (GOE) and the Gaussian unitary ensemble (GUE) in the absence of half-integer spin and other symmetries.

Universality can be observed, however, not only in the spectrum of an individual quantum system, but also in the way in which the spectrum changes due to an external perturbation. In the following we consider quantum systems that depend on an external parameter whose alteration does not change the symmetry of the system and for which the classical dynamics is chaotic for any parameter value. Correlations between the spectra of these systems at different parameter values are also found to be universal functions of the parameter difference provided that the parameter is scaled in an appropriate way [2, 3]. The universal correlation functions again agree with those of random matrix theory (RMT) and have been derived for the GOE and the GUE. The early developments of parametric spectral correlations are reviewed in [4].

One main approach to understanding the connection between quantum chaos and RMT has been the application of semiclassical methods. Spectral statistics that are related to the two-point correlation function of the density of states, like its Fourier transform, the spectral form factor $K(\tau)$, are expressed semiclassically by a double sum over the periodic orbits of the classical system. An evaluation of this double sum in the diagonal approximation, which pairs orbits with themselves or their time reverse, yields the first term in the small τ expansion of $K(\tau)$ in agreement with RMT [5, 6]. Higher order terms are due to pairs of correlated periodic orbits. In recent years there has been a rapid development of methods to evaluate orbit correlations that are responsible for the agreement with RMT. The first off-diagonal term for $K(\tau)$ was evaluated in [7, 8], and the complete small τ expansion was obtained in [9–11]. Similar methods have been applied since to derive off-diagonal terms, for example, for the conductance [12, 13], the shot noise [14] and the GOE–GUE transition [15].

Parametric spectral correlations have previously been treated within the diagonal approximation [16–18]. In this paper we extend semiclassical techniques for off-diagonal terms to include parametric correlations. For systems without time reversal symmetry we derive all terms in the small τ-expansion of the parametric spectral form factor in a closed form. In the GOE case we show that the expansion up to order τ^7 agrees with RMT. One main reason for the universal result is that in the semiclassical limit $\hbar \to 0$ the relevant quantum fluctuations are due to very small parameter variations on the classical scale. One assumption is that the parameter dependence is in some sense typical. Specifically, we assume that the derivatives of the actions of very long periodic orbits with respect to the parameter have a Gaussian distribution [17, 18]. This excludes, for example, perturbations by a point scatterer for which off-diagonal terms were calculated in [19].

In section 2 we introduce the parametric spectral form factor and in section 3 we state results of random matrix theory for it. In section 4 we consider its semiclassical approximation and the diagonal approximation, while in section 5 we derive the off-diagonal terms. For systems without time reversal symmetry the expansion is summed in section 6, and section 7 contains our conclusions.

While writing up our paper the preprint [20], which is closely related to our work, appeared on the archive. Nagao *et al* investigate parametric correlations that depend on a magnetic field difference, and obtain the universal results for the GUE case and the GOE–GUE transition by periodic orbit expansions. Our work is complementary in that we treat arbitrary parameters and consider also the GOE case.

2. The parametric spectral form factor

One way to characterize fluctuations in quantum spectra is to consider correlation functions of the density of states. For parametric correlations the density of states depends on the energy E as well as on a parameter X, and in the semiclassical regime it can be written as the sum of a mean part and an oscillatory part

$$d(E,X) = \sum_n \delta(E - E_n(X)) \approx \bar{d}(E,X) + d^{\mathrm{osc}}(E,X), \tag{1}$$

where $E_n(X)$, $n = 1, 2, \ldots$, is the nth energy level as a function of the parameter X. The mean density of states in an f-dimensional system is given by $\bar{d}(E,X) \sim \Omega(E,X)/(2\pi\hbar)^f$ in the semiclassical limit $\hbar \to 0$. $\Omega(E,X)$ is the volume of the surface of constant energy in phase space at energy E and parameter X.

In order to obtain a universal parametric spectral correlation function one has to perform two unfoldings, one in energy E and one in the parameter X of the system. A new energy parameter is defined by

$$\tilde{E} = \bar{N}(E, X), \tag{2}$$

where $\bar{N}(E, X)$ is the mean part of the spectral staircase $N(E, X) = \int_{-\infty}^{E} \mathrm{d}E' d(E', X)$. In terms of the new energy $\tilde{E}$ the density of states has a mean value of 1. The spectral statistics are evaluated in the semiclassical limit in an interval $\Delta\tilde{E}$ that is classically small but contains a large number of energy levels, i.e. it satisfies $\tilde{E} \gg \Delta\tilde{E} \gg 1$.

A new parameter $\tilde{X}$ is introduced by [2, 21]

$$\tilde{X} = \int_{X_0}^{X} \mathrm{d}X' \sigma(X'), \qquad \sigma(X') = \sqrt{\langle v_n(X')^2 \rangle}, \tag{3}$$

where $v_n(X) = \partial\tilde{E}_n/\partial X$ are the level velocities and the average is performed over the levels in the interval $\Delta\tilde{E}$. X_0 is an arbitrary parameter value at which $\tilde{X} = 0$. In terms of the new parameter $\tilde{X}$ the level velocities have a unit variance.

We may then define the unfolded two-point correlation function by

$$R_2(\eta, x) = \left\langle \tilde{d}^{\mathrm{osc}}\left(\tilde{E} + \frac{\eta}{2}, \tilde{X} + \frac{x}{2}\right) \tilde{d}^{\mathrm{osc}}\left(\tilde{E} - \frac{\eta}{2}, \tilde{X} - \frac{x}{2}\right)\right\rangle_{\tilde{E},\tilde{X}}, \tag{4}$$

where $\tilde{d}(\tilde{E}, \tilde{X})$ is the density of states of the unfolded spectrum, and the average is performed over the energy interval $\Delta\tilde{E}$ as well as over a parameter interval $\Delta\tilde{X}$. The relation to the original density of states is given by

$$\tilde{d}(\tilde{E}, \tilde{X}) = \frac{\partial N(E, X)}{\partial E} \frac{\partial E}{\partial \tilde{E}} = \frac{d(E, X)}{\bar{d}(E, X)}, \tag{5}$$

and in the semiclassical limit we find that

$$R_2(\eta, x) \sim \frac{\left\langle d^{\mathrm{osc}}\left(E + \frac{\eta}{2\bar{d}} + \frac{x\rho}{2\sigma}, X + \frac{x}{2\sigma}\right) d^{\mathrm{osc}}\left(E - \frac{\eta}{2\bar{d}} - \frac{x\rho}{2\sigma}, X - \frac{x}{2\sigma}\right)\right\rangle_{E,X}}{\bar{d}(E, X)^2}. \tag{6}$$

Equation (6) has been obtained by linearizing the unfolding equations (2) and (3), because x and η correspond to small changes on the classical scale. Explicitly, $\bar{d}$ is of order $\hbar^{-f}$ and $\sigma = \sigma(X)$ is of order $\hbar^{-(f+1)/2}$ (see equation (20) later). The term $x\rho/2\sigma$ takes account of the change of the energy when X is changed while keeping $\tilde{E}$ fixed

$$\rho = \left.\frac{\partial E}{\partial X}\right|_{\tilde{E}} = -\frac{\partial\bar{N}/\partial X}{\partial\bar{N}/\partial E}. \tag{7}$$

In the following, we will consider the parametric spectral form factor which is obtained by a Fourier transform of the parametric two-point correlation function

$$K(\tau, x) = \int_{-\infty}^{\infty} R_2(\eta, x)\, \mathrm{e}^{-2\pi \mathrm{i}\eta\tau}\, \mathrm{d}\eta. \tag{8}$$

3. Results from random matrix theory

The parametric two-point correlation function $R_2(\eta, x)$ has been derived in the context of disordered systems [2, 22] for the GUE and the GOE. For the GUE case it is given by

$$R_2^{\mathrm{GUE}}(\eta, x) = \frac{1}{2} \int_{-1}^{1} \mathrm{d}\lambda \int_{1}^{\infty} \mathrm{d}\lambda_1 \cos(\pi\eta(\lambda_1 - \lambda)) \exp\left(-\pi^2 x^2 \left(\lambda_1{}^2 - \lambda^2\right)/2\right). \tag{9}$$

After performing the Fourier transform in (8) to obtain the parametric form factor we arrive at

$$K^{\mathrm{GUE}}(\tau, x) = \frac{1}{2}\int_{-1}^{1} \mathrm{d}\lambda \int_{1}^{\infty} \mathrm{d}\lambda_1 \times \exp\left(-\pi^2 x^2 \left(\lambda_1{}^2 - \lambda^2\right)/2\right)\left[\delta(\lambda_1 - \lambda - 2\tau) + \delta(\lambda_1 - \lambda + 2\tau)\right]. \tag{10}$$

For positive τ the second delta function does not contribute because $\lambda_1 \geqslant \lambda$. From the first delta function we get the relation $2\tau = \lambda_1 - \lambda$. In the case $\tau < 1$, which we consider in the following, the domain of integration for λ_1 is reduced to $1 \leqslant \lambda_1 \leqslant 1 + 2\tau$, and we obtain

$$K^{\mathrm{GUE}}(\tau, x) = \frac{1}{2}\int_{1}^{1+2\tau} \mathrm{d}\lambda_1\, \mathrm{e}^{2\pi^2 x^2 \tau(\tau - \lambda_1)} = \frac{\sinh(2\pi^2 x^2 \tau^2)}{2\pi^2 x^2 \tau}\, \mathrm{e}^{-2\pi^2 x^2 \tau}, \qquad \tau < 1. \tag{11}$$

For comparison with the semiclassical expansion we expand the sinh function and define $B = 2\pi^2 x^2/\kappa$ where $\kappa = 1$ and 2 for the GUE and GOE cases, respectively.

$$K^{\mathrm{GUE}}(\tau, x) = \mathrm{e}^{-B\tau} \sum_{k=0}^{\infty} \frac{B^{2k}\tau^{4k+1}}{(2k+1)!}. \tag{12}$$

The parametric correlation function for the GOE case is given by a triple integral

$$R_2^{\mathrm{GOE}}(\eta, x) = \int_{-1}^{1} \mathrm{d}\lambda \int_{1}^{\infty} \mathrm{d}\lambda_1 \int_{1}^{\infty} \mathrm{d}\lambda_2 \cos(\pi\eta(\lambda - \lambda_1\lambda_2)) \frac{(1-\lambda^2)(\lambda - \lambda_1\lambda_2)^2}{\left(2\lambda\lambda_1\lambda_2 - \lambda^2 - \lambda_1{}^2 - \lambda_2{}^2 + 1\right)^2} \times \exp\left(-\pi^2 x^2 \left(2\lambda_1{}^2\lambda_2{}^2 - \lambda^2 - \lambda_1{}^2 - \lambda_2{}^2 + 1\right)/4\right). \tag{13}$$

An evaluation of the Fourier transform to obtain the parametric form factor results in a replacement of the cos term in the triple integral in (13) by a sum of the two delta-functions $\delta(\lambda - \lambda_1\lambda_2 \pm 2\tau)$. For $\tau > 0$ only the delta function with the plus sign in the argument contributes because $\lambda_1\lambda_2 \geqslant \lambda$. As we are again considering the case when $\tau < 1$ our domain of integration for the other two variables is given by $1 \leqslant \lambda_1 \leqslant 1 + 2\tau$ and $1 \leqslant \lambda_2 \leqslant \frac{1+2\tau}{\lambda_1}$. When we perform the integral over λ we are left with

$$K^{\mathrm{GOE}}(\tau, x) = \int_{1}^{1+2\tau} \mathrm{d}\lambda_1 \int_{1}^{\frac{1+2\tau}{\lambda_1}} \mathrm{d}\lambda_2 \frac{4\tau^2\left(1 - \lambda_1{}^2\lambda_2{}^2 + 4\tau\lambda_1\lambda_2 - 4\tau^2\right)}{\left(1 + \lambda_1{}^2\lambda_2{}^2 - \lambda_1{}^2 - \lambda_2{}^2 - 4\tau^2\right)^2} \times \exp\left(-\pi^2 x^2\left(1 + \lambda_1{}^2\lambda_2{}^2 - \lambda_1{}^2 - \lambda_2{}^2 + 4\tau\lambda_1\lambda_2 - 4\tau^2\right)/4\right), \tag{14}$$

for $\tau < 1$. In order to evaluate this integral as a series in τ it is useful to remove the τ dependence from the limits. This is done by changing the integration variables using $\lambda_1 = 1 + \tau y_1$ and $\lambda_1\lambda_2 = 1 + \tau y_2$. Then the expansion of the parametric form factor is obtained by expanding the integrand for small values of τ. Using Maple we performed this expansion up to seventh order and evaluated the integrals with the following result:

$$K^{\mathrm{GOE}}(\tau, x) = \mathrm{e}^{-B\tau}\left[2\tau - 2\tau^2 - (2B-2)\tau^3 + \left(2B - \frac{8}{3}\right)\tau^4 + \left(\frac{5B^2}{3} - \frac{8B}{3} + 4\right)\tau^5 - \left(\frac{5B^2}{3} - 4B + \frac{32}{5}\right)\tau^6 - \left(\frac{41B^3}{45} - \frac{11B^2}{5} + \frac{32B}{5} - \frac{32}{3}\right)\tau^7 + \cdots\right], \tag{15}$$

where B has been defined after equation (11). We extracted an exponential factor from the expansion as this is convenient for the comparison with the semiclassical result.

4. Semiclassical approximation

Our starting point is the semiclassical expression for the parametric form factor for chaotic systems that is obtained by inserting the Gutzwiller trace formula into (6) and evaluating the Fourier transform (8) in leading semiclassical order [17, 18]

$$K^{\mathrm{sc}}(\tau, x) = \frac{1}{T_H}\left\langle \sum_{\gamma,\gamma'} A_\gamma A^*_{\gamma'} \exp\left(\frac{\mathrm{i}(S_\gamma - S_{\gamma'})}{\hbar}\right) \exp\left(\frac{\mathrm{i}x(Q_\gamma + Q_{\gamma'})}{2\sigma\hbar}\right) \delta\left(T - \frac{T_\gamma + T_{\gamma'}}{2}\right)\right\rangle. \tag{16}$$

Here $T_H = 2\pi\hbar\bar{d}(E)$ is the Heisenberg time, and $\tau = T/T_H$. The double sum runs over all periodic orbits of the system with actions S_γ and amplitudes A_γ. In comparison to the spectral form factor $K(\tau)$ there is an additional dependence on the parametric velocities Q_γ,

$$Q_\gamma = \left.\frac{\partial S_\gamma}{\partial X}\right|_{\tilde{E}} = \rho\frac{\partial S_\gamma}{\partial E} + \frac{\partial S_\gamma}{\partial X}. \tag{17}$$

Before discussing the off-diagonal terms in the next section we briefly review the results for the diagonal approximation that involves only pairs of orbits which are either identical or related by time reversal

$$K^{\mathrm{diag}}(\tau, x) = \frac{\kappa}{T_H}\left\langle \sum_\gamma |A_\gamma|^2 \exp\left(\frac{\mathrm{i}xQ_\gamma}{\sigma\hbar}\right)\delta(T - T_\gamma)\right\rangle. \tag{18}$$

Here κ is 2 if the system has time reversal symmetry and 1 if it does not.

A key ingredient in the semiclassical evaluation of the parametric form factor is the distribution of the parametric velocities Q_γ in the limit of very long periodic orbits. It has been shown for chaotic systems that the Q_γ have a mean value of zero and a variance proportional to their period [16]

$$\langle Q_\gamma\rangle = 0, \qquad \langle Q_\gamma^2\rangle \sim aT, \qquad T \to \infty, \tag{19}$$

where the averages are performed over trajectories with period around T. The proportionality factor a in (19) is semiclassically related to the variance of the level velocities [21, 23]

$$\sigma^2 \sim \frac{a\kappa\bar{d}}{2\pi\hbar}. \tag{20}$$

It is generally assumed that the Q_γ have a Gaussian distribution (see e.g. [18]), and this is the main assumption that will be used in the following. Assuming in addition that the average over the Q_γ can be done independently from the actions of the orbits, one obtains

$$\left\langle \exp\left(\frac{\mathrm{i}xQ_\gamma}{\sigma\hbar}\right)\right\rangle = \exp\left(-\frac{x^2aT}{2\sigma^2\hbar^2}\right) = \mathrm{e}^{-BT/T_H}, \tag{21}$$

where $B = 2\pi^2x^2/\kappa$ has been introduced after equation (11). The remaining sum over periodic orbits is evaluated with the Hannay–Ozorio de Almeida sum rule [5]

$$\left\langle \sum_\gamma |A_\gamma|^2\delta(T - T_\gamma)\right\rangle \sim T, \qquad \text{as} \quad T \to \infty, \tag{22}$$

which results in

$$K^{\mathrm{diag}}(\tau) \sim \kappa\tau\,\mathrm{e}^{-B\tau}, \tag{23}$$

as $\hbar \to 0$. This is in agreement with the first term in the expansion of the random matrix results, (12) and (15). In the following we show how higher order terms in this expansion can be obtained from off-diagonal contributions.

5. Off-diagonal contributions

The off-diagonal terms of the parametric form factor are due to pairs of trajectories which are correlated [7, 9, 10]. In the following we briefly review the main steps in the derivation of the semiclassical expansion of the spectral form factor (in our notation $K(\tau, x = 0)$) according to [10, 11]. The correlations that are important for the expansion of the form factor for small τ are due to close self-encounters of a periodic orbit in which two or more stretches of an orbit are almost identical, possibly up to time reversal. In general, a long periodic orbit has many of these encounter regions, and they are connected by long parts of the orbit, the so-called 'loops'. The correlated pairs of orbits are almost identical along the loops, but they differ in the way in which the loops are connected in the encounter regions. Correlated orbit pairs have certain 'structures' that are characterized by the number of encounter regions V in which the loops are connected in a different way, the number of involved orbit stretches l_α in each encounter region α, and the way in which the loops are connected by these stretches. A more accurate definition of structures can be given by putting them in a one-to-one relation with permutation matrices that describe the reconnections of the loops. One defines further a vector $\boldsymbol{v}$ whose lth component, v_l, specifies the number of encounter regions with l stretches, and the total number of orbit stretches is denoted by L. Hence

$$V = \sum_{l\geqslant 2} v_l, \qquad L = \sum_\alpha l_\alpha = \sum_{l\geqslant 2} l v_l. \tag{24}$$

The semiclassical contribution to the form factor is evaluated in two steps. First the summation over orbit pairs with the same structure is evaluated by using that long periodic orbits are uniformly distributed over the surface of constant energy in phase space. Then the summation over the different structures is performed which is a combinatorial problem.

In the following we discuss a few details of this calculation. In each encounter region α with l_α orbit stretches one chooses a perpendicular Poincaré surface that is centred on one of the stretches. The relative positions of the piercings of the other stretches through the Poincaré surface are described by coordinates along the stable and unstable manifolds. The partner periodic orbit connects the loops that start and end at the encounter region in a different way and the resulting contribution to the action difference is given in the linearized approximation by

$$(\Delta S)_\alpha \approx \sum_{j=1}^{l_\alpha - 1} s_{\alpha j} u_{\alpha j}, \tag{25}$$

where $s_{\alpha j}, u_{\alpha j}, j = 1, \ldots, l_\alpha - 1$ are appropriate differences of the coordinates along the stable and unstable manifolds. For ease of notation we discuss here the two-dimensional case in which the coordinates $s_{\alpha j}$ and $u_{\alpha j}$ are scalars. If the Poincaré surface is moved along the stretches in the encounter region these coordinates decrease or increase, exponentially, however their product remains constant. The length of the encounter region is determined by requiring that all coordinates remain smaller than an arbitrary small constant c whose exact size is not relevant for the following calculations. This defines the encounter time t_{enc}^α which depends on the coordinates in the Poincaré section.

The uniform distribution of the long periodic orbits on the energy shell is then invoked to replace the sum over all orbit pairs with the same structure by an integral over the coordinates in the Poincaré surfaces. It is convenient to also sum over all structures with the same vector $\boldsymbol{v}$, because structures with the same $\boldsymbol{v}$ give the same contribution to the form factor (because the form of (27) below depends only on $\boldsymbol{v}$ and not a particular structure).

Then

$$K_{v}(\tau)=\frac{1}{T_H}\sum_{(\gamma,\gamma')}^{\text{fixed}\,\boldsymbol{v}}|A_\gamma|^2\exp(\mathrm{i}\Delta S_\gamma/\hbar)\delta(T-T_\gamma)\sim\kappa\tau N(\boldsymbol{v})$$
$$\times\int\mathrm{d}^{L-V}s\,\mathrm{d}^{L-V}u\,\frac{w_T(\boldsymbol{s},\boldsymbol{u})}{L}\exp(\mathrm{i}\boldsymbol{s}\boldsymbol{u}/\hbar),\tag{26}$$

so that $K^{\mathrm{sc}}(\tau)=K^{\mathrm{diag}}(\tau)+\sum_{\boldsymbol{v}}K_{\boldsymbol{v}}(\tau)$. $N(\boldsymbol{v})$ is the number of structures with the same $\boldsymbol{v}$, and $\boldsymbol{s}$ and $\boldsymbol{u}$ are vectors whose components are the $s_{\alpha j}$ and $u_{\alpha j}$ for all α and j. In (26) the amplitudes and periods of the two correlated orbits are set equal, and the factor $1/L$ takes care of an overcounting related to the choice of an initial point of the trajectory. $w_T(\boldsymbol{s},\boldsymbol{u})$ is the density of the self-encounters for a given structure and separation coordinates $s_{\alpha j}$ and $u_{\alpha j}$. For long orbits it is given asymptotically by

$$\frac{w_T(\boldsymbol{s},\boldsymbol{u})}{L}\sim\frac{T\left(T-\sum_\alpha l_\alpha t_{\mathrm{enc}}^\alpha\right)^{L-1}}{L!\,\Omega^{L-V}\prod_\alpha t_{\mathrm{enc}}^\alpha}.\tag{27}$$

The integrals in (26) are evaluated by using

$$\int\prod_j\mathrm{d}s_{\alpha j}\,\mathrm{d}u_{\alpha j}\left(t_{\mathrm{enc}}^\alpha\right)^k\exp\left(\mathrm{i}\sum_j s_{\alpha j}u_{\alpha j}/\hbar\right)\approx\begin{cases}0 & \text{if } k=-1 \text{ or } k\geqslant 1\\ (2\pi\hbar)^{l_\alpha-1} & \text{if } k=0.\end{cases}\tag{28}$$

Property (28) has the consequence that after expanding the numerator of (27) the only terms that survive are those that contain a product of all encounter times t_{enc}^α which is cancelled by the denominator. This allows the evaluation of the contributions of all structures, and the full expansion of the form factor is then obtained by summing up these contributions.

Let us now come back to the parametric form factor $K(\tau,x)$. As a first step we neglect differences between the periods, amplitudes and parametric velocities of the partner orbits

$$K^{\mathrm{sc}}(\tau,x)\approx\frac{1}{T_H}\left\langle\sum_{\gamma,\gamma'}|A_\gamma|^2\exp\left(\frac{\mathrm{i}(S_\gamma-S_{\gamma'})}{\hbar}\right)\exp\left(\frac{\mathrm{i}xQ_\gamma}{\sigma\hbar}\right)\delta(T-T_\gamma)\right\rangle.\tag{29}$$

In the following we consider one particular structure and evaluate the contributions of all orbit pairs with this structure to the double sum in (29). The new term that needs to be considered carefully is the exponential factor involving the parametric velocities Q_γ. Similarly as for the diagonal approximation, we want to replace this term by an average over orbit pairs of the considered structure, assuming that this average can be performed independently from the actions and amplitudes of the orbits. A direct application of (21) would not be correct. It would just yield a simple multiplicative factor $\exp(-BT/T_H)$.

The important point to note is that all the orbit pairs of a particular structure have the same number and types of encounter regions. In each encounter region there are almost identical orbit stretches, and also the changes of the action along the stretches in an encounter region are almost identical when the external parameter is varied. This has to be taken into account when performing the Gaussian average. In other words, all the considered orbit pairs have systematic correlations between different parts of the *same* periodic orbit, and one has to consider the average over the parametric velocities for the loops and encounter regions separately. One has $Q_\gamma=Q_\gamma^{\mathrm{loops}}+\sum_\alpha l_\alpha Q_\gamma^\alpha$, where Q_γ^α is the parametric velocity of one of the l_α orbit stretches in the encounter region α. Applying (21) to the change of the actions along the loops yields

$$\left\langle\exp\left(\frac{\mathrm{i}xQ_\gamma^{\mathrm{loops}}}{\sigma\hbar}\right)\right\rangle=\exp(-BT_{\mathrm{loops}}/T_H),\tag{30}$$

while the contribution from the l_α orbit stretches in the encounter region α follows as

$$\left\langle \exp\left(\frac{\mathrm{i}xl_\alpha Q_\gamma^\alpha}{\sigma\hbar}\right)\right\rangle = \exp\left(-Bl_\alpha^2 t_{\mathrm{enc}}^\alpha / T_H\right). \tag{31}$$

Combining these contributions one finds that the average over the parametric velocities is given by

$$\left\langle \exp\left(\frac{\mathrm{i}x Q_\gamma}{\sigma\hbar}\right)\right\rangle = \exp\left(-B\left(T - \sum_\alpha l_\alpha t_{\mathrm{enc}}^\alpha\right)\Big/ T_H\right) \exp\left(-B\sum_\alpha l_\alpha^2 t_{\mathrm{enc}}^\alpha / T_H\right). \tag{32}$$

As a consequence, the inclusion of the parametric velocities leads to a replacement of equation (26) by

$$\begin{aligned} K_{\boldsymbol{v}}(\tau, x) &= \frac{1}{T_H} \sum_{(\gamma,\gamma')}^{\text{fixed } \boldsymbol{v}} |A_\gamma|^2 \exp(\mathrm{i}\Delta S_\gamma/\hbar) \exp\left(\frac{\mathrm{i}x Q_\gamma}{\sigma\hbar}\right) \delta(T - T_\gamma) \\ &\sim \kappa\tau N(\boldsymbol{v}) \int \mathrm{d}^{L-V} s\, \mathrm{d}^{L-V} u \frac{z_T(\boldsymbol{s},\boldsymbol{u})}{L} \exp(\mathrm{i}\boldsymbol{s}\boldsymbol{u}/\hbar), \end{aligned} \tag{33}$$

where

$$\begin{aligned} \frac{z_T(s,u)}{L} &= \frac{w_T(s,u)}{L} \left\langle \exp\left(\frac{\mathrm{i}x Q_\gamma}{\sigma\hbar}\right)\right\rangle \\ &\sim \frac{\mathrm{e}^{-BT/T_H} T\left(T - \sum_\alpha l_\alpha t_{\mathrm{enc}}^\alpha\right)^{L-1} \prod_\alpha \exp\left(-Bl_\alpha(l_\alpha - 1)t_{\mathrm{enc}}^\alpha / T_H\right)}{L!\Omega^{L-V} \prod_\alpha t_{\mathrm{enc}}^\alpha}. \end{aligned} \tag{34}$$

The remaining steps involve an evaluation of the integrals in (33) by using (28). This is done explicitly in the following. The numerator in (34) is expanded in the encounter times and, because of (28), the only terms that contribute in the semiclassical limit are those where the encounter times in the numerator and denominator cancel exactly. As a first step we can expand the exponentials as a power series up to first order

$$\frac{z_T(s,u)}{L} \Longrightarrow \frac{\mathrm{e}^{-BT/T_H} T\left(T - \sum_\alpha l_\alpha t_{\mathrm{enc}}^\alpha\right)^{L-1} \prod_\alpha \left(1 - l_\alpha(l_\alpha - 1)Bt_{\mathrm{enc}}^\alpha / T_H\right)}{L!\Omega^{L-V} \prod_\alpha t_{\mathrm{enc}}^\alpha}. \tag{35}$$

To obtain a product of the V different encounter times in the numerator we can take r of them from the product over α and $V - r$ of them from the bracket with the exponent $L - 1$. The corresponding coefficient is obtained by combinatorial considerations. Then we sum over all values of r from 0 to V, and the result is

$$\begin{aligned} \frac{z_T(s,u)}{L} &\Longrightarrow \frac{\mathrm{e}^{-BT/T_H} T}{L!\Omega^{L-V}} \sum_{r=0}^{V} \frac{T^{L-V+r-1} B^r (L-1)!(-1)^V \prod l^{v_l}}{T_H^r (L - V - 1 + r)! r!} \\ &\quad \times \sum_{\substack{\alpha_1,\ldots,\alpha_r \\ \text{distinct}}} \left(l_{\alpha_1} - 1\right) \times \cdots \times \left(l_{\alpha_r} - 1\right). \end{aligned} \tag{36}$$

We insert this into (33), evaluate the integral with formula (28), and obtain

$$\begin{aligned} K_{\boldsymbol{v}}(\tau, x) &\sim \kappa N(\boldsymbol{v}) \frac{\mathrm{e}^{-B\tau}}{L} \sum_{r=0}^{V} \frac{\tau^{L-V+r+1} B^r (-1)^V \prod l^{v_l}}{(L - V - 1 + r)! r!} \\ &\quad \times \sum_{\substack{\alpha_1,\ldots,\alpha_r \\ \text{distinct}}} \left(l_{\alpha_1} - 1\right) \times \cdots \times \left(l_{\alpha_r} - 1\right). \end{aligned} \tag{37}$$

Equation (37) is the main result of this paper. It gives an explicit form for the semiclassical contributions to the form factor from orbit pairs that differ in encounter regions described by a

Table 1. Contribution of different types of orbit pairs to the parametric spectral form factor.

$\boldsymbol{v}$	L	V	$K_{\boldsymbol{v}}(\tau,x)/(\kappa N(\boldsymbol{v}))$	$N(\boldsymbol{v})$, no TRS	$N(\boldsymbol{v})$, TRS
$(2)^1$	2	1	$-\mathrm{e}^{-B\tau}(\tau^2+B\tau^3)$	–	1
$(2)^2$	4	2	$\mathrm{e}^{-B\tau}\left(\tau^3+B\tau^4+\frac{B^2\tau^5}{6}\right)$	1	5
$(3)^1$	3	1	$-\mathrm{e}^{-B\tau}(\tau^3+B\tau^4)$	1	4
$(2)^3$	6	3	$-\mathrm{e}^{-B\tau}\left(\frac{2\tau^4}{3}+\frac{2B\tau^5}{3}+\frac{B^2\tau^6}{6}+\frac{B^3\tau^7}{90}\right)$	–	41
$(2)^1(3)^1$	5	2	$\mathrm{e}^{-B\tau}\left(\frac{3\tau^4}{5}+\frac{3B\tau^5}{5}+\frac{B^2\tau^6}{10}\right)$	–	60
$(4)^1$	4	1	$-\mathrm{e}^{-B\tau}\left(\frac{\tau^4}{2}+\frac{B\tau^5}{2}\right)$	–	20
$(2)^4$	8	4	$\mathrm{e}^{-B\tau}\left(\frac{\tau^5}{3}+\frac{B\tau^6}{3}+\frac{B^2\tau^7}{10}+\frac{B^3\tau^8}{90}+\frac{B^4\tau^9}{2520}\right)$	21	509
$(2)^2(3)^1$	7	3	$-\mathrm{e}^{-B\tau}\left(\frac{2\tau^5}{7}+\frac{2B\tau^6}{7}+\frac{B^2\tau^7}{14}+\frac{B^3\tau^8}{210}\right)$	49	1092
$(2)^1(4)^1$	6	2	$\mathrm{e}^{-B\tau}\left(\frac{2\tau^5}{9}+\frac{2B\tau^6}{9}+\frac{B^2\tau^7}{30}\right)$	24	504
$(3)^2$	6	2	$\mathrm{e}^{-B\tau}\left(\frac{\tau^5}{4}+\frac{B\tau^6}{4}+\frac{B^2\tau^7}{20}\right)$	12	228
$(5)^1$	5	1	$-\mathrm{e}^{-B\tau}\left(\frac{\tau^5}{6}+\frac{B\tau^6}{6}\right)$	8	148

vector $\boldsymbol{v}$. The results for the different types of encounters with $L-V\leqslant 4$ are shown in table 1. The vectors $\boldsymbol{v}$ are represented in the form $(2)^{v_2}(3)^{v_3}\ldots$ and the horizontal lines separate vectors $\boldsymbol{v}$ with different value of $L-V$. The numbers $N(\boldsymbol{v})$ are the same as for the spectral form factor [11].

To find the total contribution to the form factor we now multiply the middle column that contains $K_{\boldsymbol{v}}(\tau,x)/(\kappa N(\boldsymbol{v}))$ by κ and $N(\boldsymbol{v})$, add the diagonal approximation and sum over different $\boldsymbol{v}$. If we do that for all orbits pairs with $L-V\leqslant 8$ for the case without time reversal symmetry ($\kappa=1$), we obtain the expansion for the form factor in τ up to ninth order

$$K^{\mathrm{sc}}(\tau,x)=\mathrm{e}^{-B\tau}\left[\tau+\frac{B^2\tau^5}{6}+\frac{B^4\tau^9}{120}+\cdots\right]. \tag{38}$$

This agrees with the first three terms of the expansion (12) in the section on RMT. It is noticeable that when summing over terms with the same value of $L-V$, that all terms cancel apart from the highest order term from orbit pairs with only 2-encounters. In fact we will show this using a recurrence relation in the appendix. This allows us to calculate the expansion of the form factor to all orders in τ which is done in the next section.

For systems with time reversal symmetry ($\kappa=2$) we sum over all contributions with $L-V\leqslant 6$, and obtain the expansion of the parametric form factor in τ up to seventh order

$$\begin{aligned}K^{\mathrm{sc}}(\tau,x)=\mathrm{e}^{-B\tau}\Bigg[&2\tau-2\tau^2-(2B-2)\tau^3+\left(2B-\frac{8}{3}\right)\tau^4+\left(\frac{5B^2}{3}-\frac{8B}{3}+4\right)\tau^5\\&-\left(\frac{5B^2}{3}-4B+\frac{32}{5}\right)\tau^6-\left(\frac{41B^3}{45}-\frac{11B^2}{5}+\frac{32B}{5}-\frac{32}{3}\right)\tau^7+\cdots\Bigg].\end{aligned} \tag{39}$$

This agrees with the expansion of the RMT result (15).

6. Systems without time reversal symmetry

In this section we derive the full expansion of the parametric form factor for small τ for the case of systems without time reversal symmetry. For this purpose we rewrite the expansion

$K^{\mathrm{sc}}(\tau,x) = \tau\,\mathrm{e}^{-B\tau} + \sum_{n=2}^{\infty} K_{\boldsymbol{v}}(\tau,x)$ with $K_{\boldsymbol{v}}(\tau,x)$ given in equation (37) in the following form:

$$K^{\mathrm{sc}}(\tau,x) = \tau\,\mathrm{e}^{-B\tau} + \sum_{n=2}^{\infty} \frac{\mathrm{e}^{-B\tau}}{(n-2)!} \sum_{r=0}^{n-1} S_n[f_r(\boldsymbol{v})]\tau^{n+r}B^r, \tag{40}$$

where

$$S_n[f_r(\boldsymbol{v})] = \sum_{\boldsymbol{v}}^{L-V+1=n} f_r(\boldsymbol{v})\tilde{N}(\boldsymbol{v}), \qquad \tilde{N}(\boldsymbol{v}) = \frac{N(\boldsymbol{v})(-1)^V}{L} \prod_l l^{v_l}, \tag{41}$$

and the functions $f_r(\boldsymbol{v})$ are given by

$$f_r(\boldsymbol{v}) = \frac{(L-V-1)!}{(L-V-1+r)!r!} \sum_{\substack{\alpha_1,\ldots,\alpha_r \\ \text{distinct}}} (l_{\alpha_1}-1)\times\cdots\times(l_{\alpha_r}-1). \tag{42}$$

The first two functions are $f_0(\boldsymbol{v}) = 1$ and $f_1(\boldsymbol{v}) = 1$. We need to evaluate the quantities $S_n[f_r(\boldsymbol{v})]$ for $r < n$. In the appendix A it is shown that $S_n[f_r(\boldsymbol{v})] = 0$ for $r < n-1$. Hence the only non-vanishing terms in the expansion (40) are those with $r = n-1$. Since r satisfies $r \leqslant V$ we have $V \geqslant n-1$. Together with the condition $L - V = n-1$ we find that $2V \geqslant L$. This is only satisfied for orbit pairs with V 2-encounters for which $\boldsymbol{v} = (2)^V$ and $L = 2V$. The contribution of these orbit pairs to the form factors can be calculated explicitly. We obtain from equations (41) and (42) with $r = V = n-1$, $L = 2V$, and $l_\alpha = 2$ for all α,

$$S_n[f_{n-1}(\boldsymbol{v})] = \frac{(L-V-1)!}{(L-V+n-2)!}\tilde{N}(\boldsymbol{v}) = \frac{(-1)^{n-1}2^{n-1}(n-2)!}{(2n-2)!}N(\boldsymbol{v}). \tag{43}$$

The number $N(\boldsymbol{v})$ can be obtained from an explicit formula that has been derived for systems without time reversal symmetry in [24]. In our notation it has the form

$$N(\boldsymbol{v}) = \frac{1}{L+1} \sum_{\boldsymbol{v}'\leqslant\boldsymbol{v}} \frac{(-1)^{L'-V'}L'!(L-L')!}{\prod_{k\geqslant2} k^{v_k} v_k'!(v_k-v_k')!}. \tag{44}$$

The notation $\boldsymbol{v}' \leqslant \boldsymbol{v}$ means that the sum runs over all integer vectors $\boldsymbol{v}'$ whose components satisfy $0 \leqslant v_k' \leqslant v_k$ for all k. Furthermore, $L' = L(\boldsymbol{v}')$ and $V' = V(\boldsymbol{v}')$. In the case of vectors $\boldsymbol{v}$ of the form $(2)^V$ the only non-vanishing component of $\boldsymbol{v}$ is $v_2 = V$ and the sum runs over all vectors with component $v_2' = m$ where $m = 0,\ldots,V$. The result is

$$N(\boldsymbol{v}) = \frac{1}{2n-1} \sum_{m=0}^{n-1} \frac{(-1)^m(2m)!(2n-2m-2)!}{2^{n-1}m!(n-m-1)!} = \frac{(2n-2)!}{2^{n-1}n!}\frac{1-(-1)^n}{2}, \tag{45}$$

where the last equality can be found in [25]. The expression vanishes if n is even. With n odd and $V = n-1$ we obtain $S_n[f_{n-1}(\boldsymbol{v})] = 1/n!$ and the complete expansion of the form factor is $(n = 2k+1)$

$$K^{\mathrm{sc}}(\tau,x) = \tau\,\mathrm{e}^{-B\tau} + \mathrm{e}^{-B\tau} \sum_{k=1}^{\infty} \frac{\tau^{4k+1}B^{2k}}{(2k+1)!} = \frac{\sinh(B\tau^2)}{B\tau}\,\mathrm{e}^{-B\tau} \tag{46}$$

in agreement with the RMT result (11).

7. Conclusions

This work is a continuation of recent developments in semiclassical periodic orbit expansions in chaotic systems. These methods have been applied to several spectral and transport quantities in order to demonstrate the universality of quantum fluctuation statistics of chaotic systems. We extended these ideas to include the dependence on an external parameter, and we obtained a semiclassical expansion of the parametric spectral form factor $K(\tau, x)$ for small τ in agreement with RMT. The approach of the present paper can be used to include a parameter dependence for other statistical measures as well.

The main input that is needed for the semiclassical calculation is the distribution of the parametric velocities of long orbits, which is commonly assumed to be Gaussian. For the off-diagonal terms one has to consider that the changes of the actions along the different orbit stretches in an encounter region are almost identical. This induces correlations between different parts of the *same* trajectory that have to be taken into account when performing the Gaussian average.

The limitations of the semiclassical calculations are similar to that for the spectral form factor. The method shows how to evaluate the periodic orbit correlations that are responsible for agreement with RMT, but one main open question is to show that terms that have been neglected do not contribute in leading semiclassical order. Another open point concerns the region $\tau > 1$. In this regime the random matrix expressions for $K(\tau, x)$ have a different functional form. So far, extensions to this region have relied on the diagonal approximation [17, 18].

Possible extensions of this paper include the consideration of the nonuniversal regime for very small values of τ where the Hannay–Ozorio de Almeida sum rule does not apply. Nonuniversal contributions can be obtained by including explicitly the shortest periodic orbits or, one may speculate, by setting them in relation to the eigenvalues of a classical evolution operator as for the diagonal approximation [17].

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Appendix A. Recurrence relations

In this appendix we show that the quantities $S_n[f_r(\boldsymbol{v})]$, defined in (41) and (42), vanish for $r \leqslant n-1$. The functions $f_r(\boldsymbol{v})$ are defined in terms of a restricted sum in which all summation indices are distinct. As a first step this sum is expressed by unrestricted sums. How to do this by combinatorial sieving is discussed, for example, in section 4 of [26].

We first introduce some notation. A set partition $\boldsymbol{F}$ of the set of integers $\{1, 2, \ldots, r\}$ is a decomposition of this set into disjoint subsets $[F_1, \ldots, F_\nu]$. Then $|F_1| + \cdots + |F_\nu| = r$ where $|F_i|$ is the number of elements in the set F_i. Let us define a generalization of the Kronecker delta-function

$$\delta^{\boldsymbol{F}}_{\alpha_1,\ldots,\alpha_r} = \begin{cases} 1 & \text{if } \alpha_i = \alpha_j \text{ for all } i \text{ and } j \text{ such that } i, j \in F_k \text{ for some } k, \\ 0 & \text{otherwise.} \end{cases} \tag{A.1}$$

Then

$$\sum_{\substack{\alpha_1,\ldots,\alpha_r \\ \text{distinct}}} [\cdots] = \sum_{\boldsymbol{F}} \mu(\boldsymbol{F}) \sum_{\alpha_1,\ldots,\alpha_r} \delta^{\boldsymbol{F}}_{\alpha_1,\ldots,\alpha_r} [\cdots], \tag{A.2}$$

where the first sum of the right-hand side runs over all set partitions of the set of r integers, and the corresponding Möbius function is given by

$$\mu(\boldsymbol{F}) = \prod_{i=1}^{\nu} (-1)^{|F_i|-1} (|F_i| - 1)!. \tag{A.3}$$

If we apply this to the functions $f_r(\boldsymbol{v})$ we obtain

$$f_r(\boldsymbol{v}) = \frac{(L-V-1)!}{(L-V-1+r)!r!} \sum_{\boldsymbol{F}} \mu(\boldsymbol{F}) g_{\boldsymbol{F}}(\boldsymbol{v}), \tag{A.4}$$

where

$$g_{\boldsymbol{F}}(\boldsymbol{v}) = \left(\sum_k v_k (k-1)^{|F_1|}\right) \times \cdots \times \left(\sum_k v_k (k-1)^{|F_v|}\right). \tag{A.5}$$

The expansion of the form factor $K(\tau)$ was evaluated in [9, 10] by using recurrence relations for the number of structures $N(\boldsymbol{v})$ corresponding to a vector $\boldsymbol{v}$. These recurrence relations were obtained by relating orbits with L loops to orbits with $L-1$ loops by considering all possible ways of removing a loop (i.e. letting its size shrink to zero).

For systems without time reversal symmetry the relevant recurrence relation is

$$v_2 \tilde{N}(\boldsymbol{v}) + \sum_{k\geqslant 2} v_{k+1}^{[k,2\to k+1]} k \tilde{N}(\boldsymbol{v}^{[k,2\to k+1]}) = 0. \tag{A.6}$$

Here the vector $\boldsymbol{v}^{[k,2\to k+1]}$ is obtained from the vector $\boldsymbol{v}$ by decreasing the components v_k and v_2 by 1 and increasing the component v_{k+1} by 1. Hence $L(\boldsymbol{v}^{[k,2\to k+1]}) = L(\boldsymbol{v}) - k - 2 + (k+1) = L(\boldsymbol{v}) - 1$ and $V(\boldsymbol{v}^{[k,2\to k+1]}) = V(\boldsymbol{v}) - 1$.

In order to obtain the coefficient of the form factor expansion one has to sum over the numbers $N(\boldsymbol{v})$ for all vectors for which $L(\boldsymbol{v}) - V(\boldsymbol{v}) + 1 = n$. The recurrence relation may be used for this purpose, because one can show that for each k

$$\sum_{\boldsymbol{v}}^{L-V+1=n} v_{k+1}^{[k,2\to k+1]} h(\boldsymbol{v}^{[k,2\to k+1]}) = \sum_{\boldsymbol{v}'}^{L'-V'+1=n} v'_{k+1} h(\boldsymbol{v}'), \tag{A.7}$$

where $h(\boldsymbol{v})$ is some function of $\boldsymbol{v}$. One condition is that $v_1 = v_1^{[k,2\to k+1]} = 0$, because the vectors describe encounter regions which contain at least two orbit stretches. Summing the recurrence relation (A.6) over $\boldsymbol{v}$ yields

$$0 = S_n\left[v_2 + \sum_{k\geqslant 2} v_{k+1} k\right] = S_n[L-V] = (n-1)S_n[1]. \tag{A.8}$$

This shows, for example, that all off-diagonal terms of the form factor $K(\tau, 0)$ vanish [9, 10].

We want to show in the following that $S_n[g_{\boldsymbol{F}}(\boldsymbol{v})] = 0$ if $r < n-1$. We consider first the case when the partition consists of only one subset F_1 with $|F_1| = r$. Then $g_{\boldsymbol{F}}(\boldsymbol{v}) = g_r(\boldsymbol{v})$ where

$$g_r(\boldsymbol{v}) = \sum_k v_k (k-1)^r. \tag{A.9}$$

We show that $S_n[g_r(\boldsymbol{v})] = 0$ if $r < n-1$ by induction. The statement is true for $r = 0$, because $S_n[1] = 0$ by equation (A.8). Now we fix a value of $r < n-1$ and assume that the statement is true for all smaller values of r. From the definition (A.9) follows that

$$g_r(\boldsymbol{v}^{[k,2\to k+1]}) = g_r(\boldsymbol{v}) - h_r(k), \qquad h_r(k) = (k-1)^r - k^r + 1. \tag{A.10}$$

Points that will be important in the following are that $h_r(1) = 0$ and that $h_r(k)$ is given by a finite power series in k whose highest order term is $-rk^{r-1}$.

Multiplying equation (A.6) by $g_r(\boldsymbol{v})$ and using relation (A.10) we obtain

$$0 = v_2 g_r(\boldsymbol{v}) \tilde{N}(\boldsymbol{v}) + \sum_{k\geqslant 2} v'_{k+1} k g_r(\boldsymbol{v}') \tilde{N}(\boldsymbol{v}') + \sum_{k\geqslant 2} v'_{k+1} k h_r(k) \tilde{N}(\boldsymbol{v}'), \tag{A.11}$$

where $\boldsymbol{v}' = \boldsymbol{v}^{[k,2\to k+1]}$. In the last sum we can start the sum at $k = 1$, because $h_r(1) = 0$, and then change the summation index $k \to k-1$. After summing over all vectors $\boldsymbol{v}$ we obtain

$$\begin{aligned} 0 &= S_n\left[v_2 g_r(\boldsymbol{v}) + \sum_{k\geqslant 2} v_{k+1} k g_r(\boldsymbol{v}) + \sum_{k\geqslant 2} v_k (k-1) h_r(k-1) \right] \\ &= S_n\left[(L-V) g_r(\boldsymbol{v}) - \sum_{k\geqslant 2} v_k r (k-1)^r + \cdots \right]. \end{aligned} \tag{A.12}$$

In the second line we used that $v_2 + \sum_{k\geqslant 2}^{\infty} v_{k+1} k = \sum_{l\geqslant 2} v_l (k-1) = L - V$, and we wrote only the highest order term of $h_r(k-1)$. The lower order terms, denoted by the dots, involve powers $(k-1)^m$ with $m < r$ and can be neglected due to our induction assumption. Hence we find that

$$(n-r-1) S_n[g_r(\boldsymbol{v})] = 0, \tag{A.13}$$

so that indeed $S_n[g_r(\boldsymbol{v})] = 0$ if $r < n-1$. The proof for general $g_{\boldsymbol{F}}(\boldsymbol{v})$ is very similar. We consider the general form

$$g_{\boldsymbol{F}}(\boldsymbol{v}) = \prod_{i=1}^{\nu} g_{|F_i|}(\boldsymbol{v}), \tag{A.14}$$

and we use again induction to prove that $S_n[g_{\boldsymbol{F}}] = 0$ if $r < n-1$. The statement is true for $r = 0$, and we fix a value of r and assume that it is true for all smaller values of r. In order to use the recurrence relation (A.6) we note that

$$g_{\boldsymbol{F}}(\boldsymbol{v}) = \prod_{i=1}^{\nu} \left(g_{|F_i|}(\boldsymbol{v}^{[k,2\to k+1]}) + h_{|F_i|}(k) \right). \tag{A.15}$$

We multiply equation (A.6) by $g_{\boldsymbol{F}}(\boldsymbol{v})$ and use relation (A.15) to obtain

$$\begin{aligned} 0 &= v_2 g_{\boldsymbol{F}}(\boldsymbol{v}) \tilde{N}(\boldsymbol{v}) + \sum_{k\geqslant 2} v'_{k+1} k g_{\boldsymbol{F}}(\boldsymbol{v}') \tilde{N}(\boldsymbol{v}') \\ &\quad + \sum_{k\geqslant 2} v'_{k+1} k \left[\prod_{i=1}^{\nu} \left(g_{|F_i|}(\boldsymbol{v}') + h_{|F_i|}(k) \right) - \prod_{i=1}^{\nu} g_{|F_i|}(\boldsymbol{v}') \right] \tilde{N}(\boldsymbol{v}'), \end{aligned} \tag{A.16}$$

where we added an additional term and subtracted it again. As before $\boldsymbol{v}' = \boldsymbol{v}^{[k,2\to k+1]}$. In the second sum we can start the sum at $k = 1$, because $h_i(1) = 0$ for all i, and then change the summation index $k \to k-1$. After summing over all vectors $\boldsymbol{v}$ we obtain

$$\begin{aligned} 0 &= S_n\left[v_2 g_{\boldsymbol{F}}(\boldsymbol{v}) + \sum_{k\geqslant 2} v_{k+1} k g_{\boldsymbol{F}}(\boldsymbol{v}) \right. \\ &\quad \left. + \sum_{k\geqslant 2} v_k (k-1) \left[\prod_{i=1}^{\nu} \left(g_{|F_i|}(\boldsymbol{v}) + h_{|F_i|}(k-1) \right) - \prod_{i=1}^{\nu} g_{|F_i|}(\boldsymbol{v}) \right] \right] \\ &= S_n\left[(L-V) g_{\boldsymbol{F}}(\boldsymbol{v}) + \sum_{k\geqslant 2} v_k (k-1) \sum_{j=1}^{\nu} (-|F_j| (k-1)^{|F_j|-1}) \prod_{i\neq j} g_{|F_i|}(\boldsymbol{v}) + \cdots \right]. \end{aligned} \tag{A.17}$$

In the step from the first to the second line we expanded the first product, inserted the power series for the functions $h_{|F_i|}(k-1)$ and wrote only those terms that do not vanish due to the induction assumption. We obtain further

$$0 = S_n\left[(L-V)g_{\boldsymbol{F}}(\boldsymbol{v}) - \sum_{j=1}^{\nu}|F_j|g_{|F_i|}(\boldsymbol{v})\prod_{i\neq j}g_{|F_i|}(\boldsymbol{v}) + \cdots\right]$$
$$= (n-1-r)S_n[g_{\boldsymbol{F}}(\boldsymbol{v})], \qquad \text{(A.18)}$$

which concludes the proof that $S_n[g_{\boldsymbol{F}}(\boldsymbol{v})] = 0$ for $r < n-1$.

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